TEMPERATURE FIELD OF A WEDGE-SHAPED ROD WITH A LOCAL PERIODIC SURFACE SOURCE

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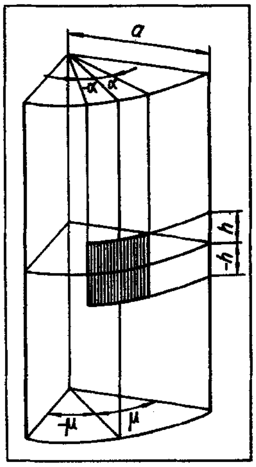
The temperature distribution in a wedge-shaped rod with a rectangular surface heat source is determined.

There have been a number of studies [1–3] of the periodic temperature fields in a rod when the entire



Wedge-shaped rod with local rectangular source.

lateral surface or end face is heated. It is necessary to calculate the fields created by periodically acting sources localized on part of the surface, for example, in connection with automatic temperature control and electrospark, electron beam, and laser techniques. This article is concerned with one such problem.

We consider an infinite rod whose lateral surfaces are two intersecting half-planes $\varphi = \pm\mu$ and part of a cylinder surface of radius $\rho = a$ (see figure). Assume that a periodically acting source $S_0 \exp(-i\omega t)$ is located on the portion $|\varphi| \le \alpha$, $|z| \le h$ of the surface $\rho = a$ and that on the rest of the surface of the rod there is no heat exchange with the ambient medium. Our problem reduces to the solution of

$$\Delta U = \frac{1}{c} U'_t,$$

in the region $0 < \rho \le a$, $|\varphi| \le \mu$, $|z| < \infty$, satisfying the boundary conditions

$$\left.\frac{\partial U}{\partial \varphi}\right|_{\varphi=\pm\mu} = 0,$$

$$\left.\frac{\partial U}{\partial \rho}\right|_{\rho=a} = \psi = \begin{cases} 0 & \text{at } |z| > h \text{ and } |\varphi| \leqslant \mu, \\ 0 & \text{at } |z| \leqslant h \text{ and } \mu \geqslant |\varphi| > \alpha, \\ S_0 \exp(-i\omega t) & \text{at } |z| \leqslant h \text{ and } |\varphi| \leqslant \alpha. \end{cases}$$

Moreover, the function U and its partial derivatives must be damped as $|z| \to \infty$.

We find a solution of the differential equation in the form $U = V \exp(-i\omega t)$. Expanding the piecewise-continuous function ψ in a Fourier series on the interval $|\varphi| \le \mu$ and omitting the multiplier $\exp(-i\omega t)$, we rewrite the differential equation and boundary conditions as

$$\Delta V + \lambda V = 0 \left(\lambda = i\frac{\omega}{c}\right), \tag{1}$$

$$\left.\frac{\partial V}{d\rho}\right|_{\rho=a} = \begin{cases} 0 \\ \sum_{n=0}^{\infty} \frac{2S_0 \sin(\nu\alpha)\cos(\nu\varphi)}{\varepsilon_n \nu\mu} \text{ when } |z| \leqslant h, \end{cases} \tag{2}$$

where $\varepsilon_0 = 2$, $\varepsilon_n = 1$ with n = 1, 2, 3, ... and $\nu = \pi n/\mu$. To solve the problem we apply a Fourier integral transformation. Multiplying the left and right sides of (1) and (2) by $\exp(i\tau z)$ and integrating with respect to the variable z from $-\infty$ to ∞ with account for damping at infinity, we obtain

$$\Delta F + \eta^2 F = 0, \tag{3}$$

$$\left.\frac{\partial F}{\partial \varphi}\right|_{\varphi=\pm\mu} = 0; \quad \left.\frac{\partial F}{\partial \rho}\right|_{\rho=a} = \frac{4S_0 \sin(\tau h)}{\mu\tau} \sum_{n=0}^{\infty} \frac{\sin(\nu\alpha)\cos(\nu\varphi)}{\varepsilon_n \nu}, \tag{4}$$

where $F = \int_{-\infty}^{\infty} V \exp(i\tau z)\,dz$ and $\eta^2 = \lambda - \tau^2$. Separating the variables in (1) and using (2), we have

$$F = \frac{4S_0}{\mu} \sum_{n=0}^{\infty} \frac{\sin(\tau h)\sin(\nu\alpha) I_\nu(\eta\rho)\cos(\nu\varphi)}{\varepsilon_n \tau\eta\nu I_\nu^1(\eta a)}. \tag{5}$$

Applying to (5) the inverse Fourier integral transformation, according to which

$$V = \frac{1}{2\pi} \int_{-\infty}^{\infty} F \exp(-i\tau z)\,d\tau,$$

we find the solution of the original problem

$$V = \frac{2S_0}{\pi} \sum_{n=0}^{\infty} \left\{ \frac{\sin(\nu\alpha)\cos(\nu\varphi)}{\nu\mu} \times \int_{-\infty}^{\infty} \frac{\sin(\tau h) I_\nu(\eta\rho)\exp(-i\tau z)\,d\tau}{\varepsilon_n \tau\eta I'_\nu(\eta a)} \right\}. \tag{6}$$

The integrand in (6) is a regular function on the entire plane of the complex variable τ with simple poles at the points

$$\tau_m^{(\nu)} = \pm\sqrt{\lambda - \left[\frac{\beta_m^{(\nu)}}{a}\right]^2},$$

where $\beta_m^{(\nu)}$ are roots of the equation $I'_\nu[\beta_m^{(\nu)}] = 0$, while the index $\nu = \pi n/\mu$ with n = 0, 1, 2, ... determines the order of the Bessel function and its derivative. The points $\tau_0^{(0)} = \pm(\lambda)^{1/2}$ are first-order poles only for n = 0, and the integrand tends exponentially to zero when $\tau \to \infty$, remaining in the lower half-plane ($I_m\tau < 0$) at $z > h$ or in the upper half-plane ($I_m\tau > 0$) at $z < -h$.

Setting $z > h$, we find the residue of the integrand at the pole $\tau_0^{(0)} = -(\lambda)^{1/2}$,

$$\frac{ic\sin(h\sqrt{\lambda})\exp(-i\sqrt{\lambda}z)}{2a\omega} \qquad (\beta_0^{(0)} = 0)$$

and at the pole $\tau_m^{(\nu)}$ ($I_m\tau_m^{(\nu)} < 0$, n = 0, 1, 2, ...)

$$\frac{\sin[\tau_m^{(\nu)}h]I_\nu\left[\rho\frac{\beta_m^{(\nu)}}{a}\right]\exp[-i\tau_m^{(\nu)}z]}{-a[\tau_m^{(\nu)}]^2I''_\nu[\beta_m^{(\nu)}]}.$$

Multiplying the sum of the residues for poles lying in the lower half-plane by $2\pi i$ we find the value of the integral (Jordan's lemma [4]). Expression (6) reduces to

$$V(\rho,\varphi,z) = -\frac{2\alpha S_0c\sin(h\sqrt{\lambda})\exp(-i\sqrt{\lambda}|z|)}{a\mu\omega} + \frac{4S_0}{i\mu a}\sum_{n=0}^{\infty}\left\{\frac{\sin(\nu\alpha)\cos(\nu\varphi)}{\varepsilon_n\nu}\times\sum_{m=1}^{\infty}\frac{\sin[\tau_m^{(\nu)}h]I_\nu\left[\frac{\rho\beta_m^{(\nu)}}{a}\right]\exp[-i\tau_m^{(\nu)}|z|]}{[\tau_m^{(\nu)}]^2I''_\nu[\beta_m^{(\nu)}]}\right\}. \quad (7)$$

In the particular case when $\mu = \pi$, $\nu = n$ the wedge-shaped rod degenerates into a cylinder of radius $\rho = a$, with a rectangular source on the lateral surface.

For $\mu = \alpha = \pi$, i.e., for a rod with an annular heat source, (7) takes its simplest form

$$V = -\frac{2S_0c\sin(h\sqrt{\lambda})\exp(-i\sqrt{\lambda}|z|)}{a\omega} + \frac{2S_0i}{a}\times\sum_{m=1}^{\infty}\frac{\sin[\tau_m^{(0)}h]I_0\left[\frac{\rho\beta_m^{(0)}}{a}\right]\exp[-i\tau_m^{(0)}|z|]}{[\tau_m^{(0)}]^2I'_1[\beta_m^{(0)}]}. \quad (8)$$

From (7) and (8) it follows that in rods of the type considered there is a superposition of thermal waves whose amplitudes decrease exponentially with increase in the z-coordinate and in the parameters n and m.

NOTATION

ρ, φ, and z are cylindrical coordinates; $\pm\mu$ is the angle determining the position of the two lateral surfaces of the rod; a is the radius of the cylindrical surface of the rod; 2h is the linear dimension of the heat source along the rod axis; 2α is the angular dimension of the source; S_0 is the heat flux amplitude at the rod surface; ω is the frequency; t is the time; V is the amplitude of the temperature field; τ is the Fourier transform parameter; I_ν is the Bessel function; ν is the index determining the order of the Bessel function; and $\beta_m^{(\nu)}$ is the root of the function I'_ν.

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